ON THE EQUATIONS OF MOTION OF NONHOLONOMIC MECHANICAL SYSTEMS IN POINCARÉ-CHETAEV VARIABLES

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The equations of motion of nonholonomic systems in Poincaré-Chetaev variables [1] are derived directly from the general equation of dynamics with simultaneous allowance for all imposed constraints. Their equivalence to equations of motion derived by other methods is proved.

1. The equations of motion of nonholonomic systems. Let us consider a nonholonomic system with l degrees of freedom whose positions are defined by n Poincaré-Chetaev variables $x_1, \ldots, x_n$ with p holonomic and q nonholonomic linear constraints.

As in [1], let $X_0, X_1, \ldots, X_k$ with the commutators

$$(X_r, X_s) = \sum_{t=1}^{k} C_{rst} X_t \qquad (r = 0, 1, \ldots, k;\ s = 1, \ldots, k;\ k = n - p) \tag{1.1}$$

be the displacement operators of the so-called associated holonomic system obtained by removing all q nonholonomic constraints from the system under consideration; $\eta_1, \ldots, \eta_k$ and $\omega_1, \ldots, \omega_k$ are the parameters of the real and possible displacements of this system; k is the number of degrees of freedom; the nonholonomic constraints are reducible to the relations

$$\eta_\nu = \sum_{s=1}^{l} c_{\nu s}\eta_s + c_{\nu 0}, \quad \omega_\nu = \sum_{s=1}^{l} c_{\nu s}\omega_s \qquad (\nu = l + 1, \ldots, k;\ l = k - q) \tag{1.2}$$

Here C_{rst}, $c_{\nu s}$, $c_{\nu 0}$ are certain functions of t and x_i which depend only on the constraint conditions and on the choice of the parameters η_s and ω_s of the corresponding holonomic system.

Then, by virtue of (2.2) of [1] and (1.2), the changes df and δf in an arbitrary function $f(t, x_i)$ on the real and possible displacements of the nonholonomic system, when all $p + q$ constraint conditions are fulfilled, are given by Formulas

$$df = \Big[Y_0(f) + \sum_{s=1}^{l} \eta_s Y_s(f)\Big] dt, \qquad \delta f = \sum_{s=1}^{l} \omega_s Y_s(f) \tag{1.3}$$

Here $Y_0, Y_1, \ldots, Y_l$ are the displacement operators of the nonholonomic system, which can be expressed in terms of the operators X_s and the commutators, i.e.

$$Y_s = X_s + \sum_{\nu=l+1}^{k} c_{\nu s} X_\nu, \quad (Y_r, Y_s) = \sum_{t=1}^{l} k_{rst} Y_t + \sum_{\nu=l+1}^{k} k^*_{rs\nu} X_\nu \ \begin{pmatrix} r = 0, 1, \ldots, l \\ s = 1, \ldots, l \end{pmatrix} \tag{1.4}$$

The coefficients in Expression (1.4) are given by Formulas

$$k_{rst} = C_{rst} + \sum_{\nu=l+1} c_{\nu s} C_{r\nu t} + \sum_{\mu=l+1}^{k} c_{\mu r} \Big(C_{\mu st} + \sum_{\nu=l+1}^{k} c_{\nu s} C_{\mu\nu t}\Big)$$

$$k^*_{rs\nu} = k_{rs\nu} - \sum_{t=1}^{l} c_{\nu t} k_{rst} + Y_r(c_{\nu s}) - Y_s(c_{\nu r}) \tag{1.5}$$

$$\begin{pmatrix} r = 0, 1, \ldots, l; \ s = 1, \ldots, l \\ t = 1, \ldots, l, \ l+1, \ldots, k; \ \nu = l+1, \ldots k \end{pmatrix}$$

We can use operators (1.4) to find the equations of motion of a nonholonomic system with ideal constraints and the force function U from the general equation of dynamics

$$\sum_{i=1}^{N} \left[\left(m_i u_i'' - \frac{\partial U}{\partial u_i} \right) \delta u_i + \left(m_i v_i'' - \frac{\partial U}{\partial v_i} \right) \delta v_i + \left(m_i w_i'' - \frac{\partial U}{\partial w_i} \right) \delta w_i \right] = 0 \quad (1.6)$$

Here N denotes the number of material points of the system; u_i, v_i, w_i are the Cartesian coordinates of the ith point, which, by the conditions of the problem, are functions of the variables t, $x_1, \ldots, x_n$; u_i'', v_i'', w_i'' are its accelerations; δu_i, δv_i, δw_i are the possible displacements of a point permitted by all the constraints, and defined, in accordance with (1.3), by Formulas

$$\delta u_i = \sum_{s=1}^{l} \omega_s Y_s(u_i), \qquad \delta v_i = \sum_{s=1}^{l} \omega_s Y_s(v_i), \qquad \delta w_i = \sum_{s=1}^{l} \omega_s Y_s(w_i) \qquad (i = 1, \ldots, N) \quad (1.7)$$

To this end, substituting (1.7) into (1.6), by virtue of the independence of $\omega_1, \ldots, \omega_l$, we obtain

$$\sum_{i=1}^{N} m_i \left[u_i'' Y_s(u_i) + v_i'' Y_s(v_i) + w_i'' Y_s(w_i) \right] - Y_s(U) = 0 \quad (s = 1, \ldots, l) \quad (1.8)$$

or

$$\frac{d}{dt} \sum_{i=1}^{N} m_i \left[u_i' Y_s(u_i) + v_i' Y_s(v_i) + w_i' Y_s(w_i) \right] - Y_s(U) -$$
$$- \sum_{i=1}^{N} m_i \left[u_i' \frac{dY_s(u_i)}{dt} + v_i' \frac{dY_s(v_i)}{dt} + w_i' \frac{dY_s(w_i)}{dt} \right] = 0 \quad (s = 1, \ldots, l) \quad (1.9)$$

Here u_i', v_i', w_i' are the velocities of the point defined, according to (1.3), by formulas written only for $f = u_i$,

$$u_i' = Y_0(u_i) + \sum_{s=1}^{l} \eta_s Y_s(u_i) \qquad (i = 1, \ldots, N) \quad (1.10)$$

From (1.10) we obtain

$$Y_s(u_i) = \frac{\partial u_i'}{\partial \eta_s}, \quad Y_s(v_i) = \frac{\partial v_i'}{\partial \eta_s}, \quad Y_s(w_i) = \frac{\partial w_i'}{\partial \eta_s} \qquad \begin{pmatrix} s = 1, \ldots, l \\ i = 1, \ldots, N \end{pmatrix} \quad (1.11)$$

and, according to (1.3), for the functions $f = u_i, v_i, w_i$ we have

$$\frac{dY_s(f)}{dt} = Y_s\left(\frac{df}{dt}\right) + (Y_0, Y_s) f + \sum_{r=1}^{l} \eta_s (Y_r, Y_s) f \quad (s = 1, \ldots, l) \quad (1.12)$$

Substituting (1.11) and (1.12) into (1.9) with allowance for (1.4), we obtain

$$\frac{d}{dt} \frac{\partial T}{\partial \eta_s} - Y_s(T + U) - \sum_{t=1}^{l} \left(k_{0st} + \sum_{r=1}^{l} \eta_r k_{rst} \right) \frac{\partial}{\partial \eta_t} -$$
$$- \sum_{\nu=l+1}^{k} \left(k_{0s\nu}^* + \sum_{r=1}^{l} \eta_r k_{rs\nu}^* \right) \left(\frac{\partial T^\circ}{\partial \eta_\nu} \right) = 0 \qquad (s = 1, \ldots, l) \qquad (1.13)$$

These are the equations of motion of a nonholonomic system in Poincaré-Chetaev variables derived from the general equation of dynamics with simultaneous allowance for all the constraints imposed on the system beginning at the initial instant. Here T is the kinetic energy of the nonholonomic system,

$$T(t, x_i, \eta_1, \ldots, \eta_l) = \frac{1}{2}\sum_{i=1}^{N} m_i (u_i'^2 + v_i'^2 + w_i'^2) \qquad (1.14)$$

and $(\partial T^\circ/\partial\eta_\nu)$ denote the Expressions

$$\left(\frac{\partial T^\circ}{\partial\eta_\nu}\right) = \sum_{i=1}^{N} m_i [u_i' X_\nu(u_i) + v_i' X_\nu(v_i) + w_i' X_\nu(w_i)] \qquad (1.15)$$

$$(\nu = l+1, \ldots, k)$$

Let us show that the $(\partial T^\circ/\partial\eta_\nu)$ have the mechanical values of the impulses corresponding to the dependent parameters η_ν of (1.2) if T° is the kinetic energy of the corresponding holonomic system computed without allowance for the nonholonomic constraints or for (1.2) using Formulas

$$T^\circ = \frac{1}{2}\sum_{i=1}^{N} m_i (u_i'^2 + v_i'^2 + w_i'^2), \qquad u_i = X_0(u_i) + \sum_{s=1}^{k} \eta_s X_s(u_i) \quad (i = 1, \ldots, N) \qquad (1.16)$$

Here the formulas for the derivatives have been written for u_i' only. In fact, from (1.16) we can obtain, among other things, Expressions

$$X_\nu(u_i) = \frac{\partial u_i'}{\partial\eta_\nu}, \qquad X_\nu(v_i) = \frac{\partial v_i'}{\partial\eta_\nu}, \qquad X_\nu(w_i) = \frac{\partial w_i'}{\partial\eta_\nu} \qquad \begin{pmatrix}\nu = l+1, \ldots, k \\ i = 1, \ldots, N\end{pmatrix} \qquad (1.17)$$

Substituting (1.17) into (1.15), we obtain from these relations expressions for the impulses $\partial T^\circ/\partial\eta_\nu$.

Eqs. (1.13) coincide with Eqs. (3.14) of [1] obtained by the Chaplygin method [2], since the kinetic energy T in (1.13) computed from Formula (1.14), and the function Θ in [1] are equal to each other. This can be verified by computation.

2. The equivalence of the equations of motion of nonholonomic systems. There are at present many methods for deriving the equations of motion of nonholonomic systems. This often raises the question of their equivalence [3]. In this connection, having shown that the above direct method and the method of Chaplygin yield equivalent results, let us consider the methods of Appell [4], Hamel [5], Volterra [6], et al.

Differentiating (1.10) with respect to t, we obtain

$$u_i'' = \sum_{s=1}^{l} \eta_s' Y_s(u_i) + \ldots, \quad v_i'' = \sum_{s=1}^{l} \eta_s' Y_s(v_i) + \ldots,$$

$$w_i'' = \sum_{s=1}^{l} \eta_s' Y_s(w_i) + \ldots \quad (i = 1, \ldots, N) \qquad (2.1)$$

Here we have written only the formulas for u_i''; the dotted lines denote terms not containing $\eta_s' = d\eta_s/dt$ $(s = 1, \ldots, l)$.

Stipulating that

$$Y_s(u_i) = \frac{\partial u_i''}{\partial\eta_s'}, \quad Y_s(v_i) = \frac{\partial v_i''}{\partial\eta_s'}, \quad Y_s(w_i) = \frac{\partial w_i''}{\partial\eta_s'} \qquad \begin{pmatrix}s = 1, \ldots, l \\ i = 1, \ldots, N\end{pmatrix} \qquad (2.2)$$

and substituting these quantities into (1.8), we obtain from (2.1) the Appell equations for a nonholonomic system in Poincaré-Chetaev variables,

$$\frac{\partial S}{\partial\eta_s'} = Y_s(U) \qquad (s = 1, \ldots, l) \qquad (2.3)$$

Here S is the acceleration energy computed from Formulas (2.1),

$$S = \frac{1}{2}\sum_{i=1}^{N} m_i (u_i''^2 + v_i''^2 + w_i''^2) \tag{2.4}$$

The Appell Eqs. (2.3) are equivalent to Eqs. (1.13), since computations show that

$$\frac{\partial S}{\partial \eta_s'} = \frac{d}{dt}\frac{\partial T}{\partial \eta_s} - \sum_{i=1}^{N} m_i \left[u_i' \frac{dY_s(u_i)}{dt} + v_i' \frac{dY_s(v_i)}{dt} + w_i' \frac{dY_s(w_i)}{dt} \right] \qquad (s = 1, \ldots, l) \tag{2.5}$$

Moreover, the right sides of (2.5) become (1.13), as we can see (*) from (1.9).

According to Hamel [5], the equations of motion of nonholonomic systems are derivable from the conditions

$$\frac{d\delta u_i}{dt} - \delta u_i' = 0, \quad \frac{d\delta v_i}{dt} - \delta v_i' = 0, \quad \frac{d\delta w_i}{dt} - \delta w_i' = 0 \qquad (i = 1, \ldots, N) \tag{2.6}$$

established for all Cartesian coordinates of points of the system (**), and from the Beltrami equation, which can be written in Poincaré-Chetaev variables as

$$\frac{d}{dt}\sum_{s=1}^{k} \omega_s \frac{\partial T^\circ}{\partial \eta_s} - \delta (T^\circ + U) = 0 \tag{2.7}$$

The operations d and δ are employed here without allowance for nonholonomic constraints, i.e. according to (1.5) of [1]; T° is the kinetic energy of the corresponding holonomic system computed from Formulas (1.16). In Hamel's method nonholonomic constraints (1.2) are allowed for only after (2.7) under (2.6) has already been reduced to form (3.2) of [1]. This implies the natural equivalence of (1.13) and the equations obtained by Hamel's method, since the latter can be reduced to (1.13) or to the equivalent Eqs. (3.14) of [1] by converting to the kinetic energy T according to Formula (3.8) of [1].

For example, let us consider the case $c_{\nu s} = c_{\nu 0} = 0$ or $\eta_\nu = 0$, $\omega_\nu = 0$ (see [5 and 3]). The Hamel equations are then

$$\frac{d}{dt}\left(\frac{\partial T^\circ}{\partial \eta_s}\right)_{\eta_\nu = 0} - [X_s (T^\circ + U)]_{\eta_\nu = 0} - \sum_{t=1}^{k} \left(C_{0st} + \sum_{r=1}^{l} \eta_r C_{rst}\right) \left(\frac{\partial T^\circ}{\partial \eta_t}\right)_{\eta_\nu = 0} = 0 \qquad (s = 1, \ldots, l) \tag{2.8}$$

Converting to the kinetic energy T or Θ according to Formula (3.8) of [1] and making the substitutions

$$\left(\frac{\partial T^\circ}{\partial \eta_s}\right)_{\eta_\nu = 0} = \frac{\partial T}{\partial \eta_s}, \quad [X_s (T^\circ)]_{\eta_\nu = 0} = X_s(T) \qquad (s = 1, \ldots, l) \tag{2.9}$$

in (2.8), we obtain the special case of Eqs. (1.13) when $c_{\nu s} = c_{\nu 0} = 0$

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - X_s (T + U) - \sum_{t=1}^{l} \left(C_{0st} + \sum_{r=1}^{l} \eta_r C_{rst}\right) \frac{\partial T}{\partial \eta_t} - \sum_{\nu = l+1}^{k} \left(C_{0s\nu} + \sum_{r=1}^{l} \eta_r C_{rs\nu}\right) \left(\frac{\partial T^\circ}{\partial \eta_\nu}\right) = 0 \qquad (s = 1, \ldots, l) \tag{2.10}$$

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*) The problem of equivalence of the Chaplygin [2] and Appell [4] equations is discussed by M. I. Efimov in his candidate's thesis "On Chaplygin's Equations for Nonholonomic Systems" (Institute of Mechanics, Akad. Nauk SSSR) and by Shagi-Sultan in [7].

**) These conditions are justified for nonholonomic systems in [5, 6 and 8 — 10].

In [6] Volterra derived the equations of motion of nonholonomic systems from the Beltrami equation and conditions (2.6), defining the operations d and δ as in (1.3),

$$\sum_{s=1}^{l} \omega_s \left[\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - Y_s(T+U)\right] + \sum_{t=1}^{l}\left(\frac{d\omega_t}{dt} - \delta\eta_t\right)\frac{\partial T}{\partial \eta_t} = 0 \qquad (2.11)$$

$$\sum_{t=1}^{l}\left(\frac{d\omega_t}{dt} - \delta\eta_t\right) Y_t(u_i) = -\sum_{s=1}^{l}\omega_s\Big[\sum_{t=1}^{l}\Big(k_{0st} + \sum_{r=1}^{l}\eta_r k_{rst}\Big)Y_t(u_i) +$$

$$+\sum_{\nu=l+1}^{k}\Big(k_{0s\nu}^{*} + \sum_{r=1}^{l}\eta_r k_{rs\nu}^{*}\Big)X_\nu(u_i)\Big] \qquad (i=1,\ldots,N) \qquad (2.12)$$

Here T is kinetic energy (1.14); Expressions (2.12) for v_i and w_i are obtainable in similar fashion. Following Volterra, we multiply (2.12) by $m_i Y_s(u_i)$, $m_i Y_s(v_i)$, $m_i Y_s(w_i)$, sum over i from 1 to N, and solve the result for $d\omega_l/dt - \delta\eta_l$

$$\frac{d\omega_t}{dt} - \delta\eta_t = -\sum_{s=1}^{l}\omega_s\Big(a_{0st} + \sum_{r=1}^{l}\eta_r a_{rst}\Big) \qquad (t=1,\ldots,l) \qquad (2.13)$$

Substituting (2.13) into (2.11), we obtain the Volterra equations in Poincaré-Chetaev variables

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - Y_s(T+U) - \sum_{t=1}^{l}\Big(a_{0st} + \sum_{r=1}^{l}\eta_r a_{rst}\Big)\frac{\partial T}{\partial \eta_t} = 0 \qquad (s=1,\ldots,l) \qquad (2.14)$$

Here

$$a_{rst} = k_{rst} + \sum_{\nu=l+1}^{k} k_{rs\nu}^{*}\sum_{k=1}^{l} a_{kt}^{-1}\sum_{i=1}^{N} m_i\,[Y_k(u_i)X_\nu(u_i) +$$

$$+ Y_k(v_i)X_\nu(v_i) + Y_k(w_i)X_\nu(w_i)] \qquad \begin{pmatrix} r=0,1,\ldots,l \\ s,\ t=1,\ldots,l\end{pmatrix} \qquad (2.15)$$

where a_{kl}^{-1} is an element of the inverse of the matrix whose elements are the coefficients a_{kt} of the products $\eta_k\eta_t$ in the quadratic part of the kinetic energy T (1.14).

The above derivation of Eqs. (2.14) cannot be considered adequately justifiable for nonholonomic systems, since, generally speaking, (2.13) may not be a solution of system (2.12) because of its indeterminacy (this was noted in [6 and 3]). In fact, substituting (2.13) into (2.12), we obtain the following expression for u_j (and analogously for v_j and w_j):

$$X_\nu(u_j) = \sum_{t=1}^{l} Y_t(u_j)\sum_{k=1}^{l} a_{kt}^{-1}\sum_{i=1}^{N} m_i\,[Y_k(u_i)X_\nu(u_i) +$$

$$+ Y_k(v_i)X_\nu(v_i) + Y_k(w_i)X_\nu(w_i)] \qquad \begin{pmatrix}\nu=l+1,\ldots,k \\ j=1,\ldots,N\end{pmatrix} \qquad (2.16)$$

These conditions are not fulfilled, for example, in the case of a hoop (see Section 3).

Nevertheless, Eqs. (2.14) are valid for nonholonomic systems (this fact was noted in [11]), since, despite nonfulfillment of (2.16), the operation for solving (2.12) by the Volterra method and the operation of multiplying (2.13) by $\partial T/\partial\eta_t$ are relatively inverse for Eq. (2.11).

The validity of Eqs. (2.14) can also be verified as follows. Substituting (1.10) into

$$\frac{\partial T}{\partial \eta_t} = \sum_{i=1}^{N} m_i\,[u_i' Y_t(u_i) + v_i' Y_t(v_i) + w_i' Y_t(w_i)] \qquad (t=1,\ldots,l) \qquad (2.17)$$

and solving them for η_k, we obtain

$$\eta_k = \sum_{t=1}^{l} a_{kt}^{-1} \frac{\partial T}{\partial \eta_t} - \sum_{t=1}^{l} a_{kt}^{-1} a_{0t} \qquad (k = 1, \ldots, l) \tag{2.18}$$

Here a_{0t} are the coefficients of η_t in the linear part of the kinetic energy (1.14). On substituting (1.10) and (2.18) into Expressions (1.15), we obtain

$$\left(\frac{\partial T^\circ}{\partial \eta_\nu}\right) = \sum_{t=1}^{l} \frac{\partial T}{\partial \eta_t} \sum_{k=1}^{l} a_{kt}^{-1} \sum_{i=1}^{N} m_i [Y_k(u_i) X_\nu(u_i) +$$

$$+ Y_k(v_i) X_\nu(v_i) + Y_k(w_i) X_\nu(w_i)] \qquad (\nu = l+1, \ldots, k) \tag{2.19}$$

By virtue of (2.19), Eqs. (1.13) in notation (2.15) coincide with the Volterra equations (2.14), which proves their validity for nonholonomic systems.

The equations of Ferrers [12] for a nonholonomic system with l degrees of freedom defined by $3N$ Cartesian coordinates x_i, y_i, z_i, subject to smooth nonholonomic constraints by virtue of which the velocities x_i', y_i', z_i' can be expressed in terms of some l unknowns $\theta_1', \ldots, \theta_l'$,

$$x_i' = \sum_{s=1}^{l} a_{is}\theta_s', \qquad y_i' = \sum_{s=1}^{l} b_{is}\theta_s', \qquad z_i' = \sum_{s=1}^{l} c_{is}\theta_s' \qquad (i = 1, \ldots, N) \tag{2.20}$$

are of the form (*)

$$\frac{d}{dt}\frac{\partial T}{\partial \theta_s'} - \sum_{i=1}^{N} m_i (x_i' a_{is}' + y_i' b_{is}' + z_i' c_{is}') = \frac{\partial U}{\partial \theta_s} \qquad (s = 1, \ldots, l) \tag{2.21}$$

Here a_{is}', b_{is}', c_{is}' are the derivatives of a_{is}, b_{is}, c_{is} with respect to t; $\partial/\partial\theta_s$ are the operators

$$\frac{\partial}{\partial \theta_s} = \sum_{i=1}^{N} \left(a_{is}\frac{\partial}{\partial x_i} + b_{is}\frac{\partial}{\partial y_i} + c_{is}\frac{\partial}{\partial z_i}\right) \qquad (s = 1, \ldots, l) \tag{2.22}$$

By virtue of the fact

$$a_{is}' = \sum_{r=1}^{l} \frac{\partial a_{is}}{\partial \theta_r}\theta_r', \qquad b_{is}' = \sum_{r=1}^{l} \frac{\partial b_{is}}{\partial \theta_r}\theta_r', \qquad c_{is}' = \sum_{r=1}^{l} \frac{\partial c_{is}}{\partial \theta_r}\theta_r' \tag{2.23}$$

$$\frac{\partial x_i'}{\partial \theta_s} = \sum_{r=1}^{l} \frac{\partial a_{ir}}{\partial \theta_s}\theta_r' \qquad \frac{\partial y_i'}{\partial \theta_s} = \sum_{r=1}^{l} \frac{\partial b_{ir}}{\partial \theta_s}\theta_r', \qquad \frac{\partial z_i'}{\partial \theta_s} = \sum_{r=1}^{l} \frac{\partial c_{ir}}{\partial \theta_s}\theta_r'$$

$$(r, s = 1, \ldots, l;\ i = 1, \ldots, N)$$

Eqs. (2.21) can be reduced to the form of (1.13),

$$\frac{d}{dt}\frac{\partial T}{\partial \theta_s'} - \frac{\partial (T+U)}{d\theta_s} - \sum_{r=1}^{l} \theta_r' \sum_{i=1}^{N} \left[\left(\frac{\partial a_{is}}{\partial \theta_r} - \frac{\partial a_{ir}}{\partial \theta_s}\right)\frac{\partial T^\circ}{\partial x_i'} +\right.$$

$$\left. + \left(\frac{\partial b_{is}}{\partial \theta_r} - \frac{\partial b_{ir}}{\partial \theta_s}\right)\frac{\partial T^\circ}{\partial y_i'} + \left(\frac{\partial c_{is}}{\partial \theta_r} - \frac{\partial c_{ir}}{\partial \theta_s}\right)\frac{\partial T^\circ}{\partial z_i'}\right] = 0 \qquad (s = 1, \ldots, l) \tag{2.24}$$

$$T^\circ = \frac{1}{2}\sum_{i=1}^{N} m_i (x_i'^2 + y_i'^2 + z_i'^2)$$

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*) In [13] Appell investigated the case where θ_s are the true generalized coordinates of the system.

Eqs. (2.24) in generalized coordinates were obtained in [14]. They can be generalized for the case of Poincaré-Chetaev variables by considering a nonholonomic system defined by n variables $x_1,\ldots, x_n$ with $n-k$ holonomic constraints used to construct system (1.1) with $k-l$ nonholonomic constraints, by virtue of which the parameters $\eta_1,\ldots,\eta_k$ and $\omega_1,\ldots,\omega_k$ can be expressed in terms of l independent quantities θ_s' $\delta\theta_s$ in the form

$$\eta_\nu = \sum_{s=1}^{l} c_{\nu s}\theta_s' + c_{\nu 0}, \qquad \omega_\nu = \sum_{s=1}^{l} c_{\nu s}\delta\theta_s \qquad (\nu = 1,\ldots,k) \qquad (2.25)$$

Now, taking θ_s', $\delta\theta_s$ as the parameters of the real and possible displacements of the nonholonomic system, instead of (1.5), (1.7) and (1.13), we obtain

$$Y_0 = X_0 + \sum_{\nu=1}^{k} c_{\nu 0}X_\nu, \qquad Y_s = \sum_{\nu=1}^{k} c_{\nu s}X_\nu \qquad (s = 1,\ldots,l) \qquad (2.26)$$

$$k_{rs\nu} = \sum_{\mu=1}^{k} c_{\mu s}\left(C_{r\mu\nu} + \sum_{\gamma=1}^{k} c_{\gamma r}C_{\gamma\mu\nu}\right), \qquad k^*_{rs\nu} = k_{rs\nu} + Y_r(c_{\nu s}) + Y_s(c_{\nu r}) \qquad (2.27)$$

$$(r = 0, 1,\ldots,l;\ s = 1,\ldots,l;\ \nu = 1,\ldots,k)$$

$$\frac{d}{dt}\frac{\partial T}{\partial\theta_s'} - Y_s(T+U) - \sum_{\nu=1}^{k}\left(k^*_{0s\nu} + \sum_{r=1}^{l}\theta_r' k^*_{rs\nu}\right)\left(\frac{\partial T^\circ}{\partial\eta_\nu}\right) = 0 \quad (s = 1,\ldots,l) \qquad (2.28)$$

Here T° is the kinetic energy computed from Formulas (1.16).

Eqs. (2.28) subsume as special cases Eqs. (2.24) in Cartesian and generalized coordinates. When (2.25) are of the form (1.2), Eqs. (2.28) can be reduced to the form of Eqs. (1.13), which implies their equivalence.

3. Example. Let us consider the motions of a hoop defined by the six variables $\theta, \psi, \varphi, \xi, \eta, \zeta$ under the holonomic constraint [13]

$$\zeta - a\sin\theta = 0 \qquad (3.1)$$

and the nonholonomic constraints

$$\xi' - a\sin\psi\sin\theta\theta' + a\cos\psi\cos\theta\psi' + a\cos\psi\varphi' = 0$$

$$\eta' + a\cos\psi\sin\theta\theta' + a\sin\psi\cos\theta \quad + a\sin\psi\varphi' = 0 \qquad (3.2)$$

Taking $\theta, \psi, \varphi, \xi, \eta, \zeta$ as the Poincaré-Chetaev parameters, and the projections p, q, r, of the angular velocity (defined in [13]) and ξ', η' as the parameters of the true displacements of the holonomic system corresponding to the hoop (without allowance for constraints (3.2)), we obtain

$$\eta_1 = p = \theta', \quad \eta_2 = q = \psi'\sin\theta, \quad \eta_3 = r = \psi'\cos\theta + \varphi', \quad \eta_4 = \xi', \quad \eta_5 = \eta' \quad (3.3)$$

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial\theta} + a\cos\theta\frac{\partial}{\partial\zeta}, \quad X_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}$$

$$X_3 = \frac{\partial}{\partial\varphi}, \quad X_4 = \frac{\partial}{\partial\xi}, \quad X_5 = \frac{\partial}{\partial\eta} \qquad (3.4)$$

The commutators of operators (3.4) with the exception of

$$(X_1, X_2) = -\operatorname{ctg}\theta\ X_2 + X_3$$

are equal to zero.

Nonholonomic constraints (3.2) reduce to (1.2) in the form

$$\eta_4 = a\sin\psi\sin\theta\eta_1 - a\cos\psi\eta_3, \qquad \eta_5 = -a\cos\psi\sin\theta\eta_1 - a\sin\psi\eta_3 \qquad (3.5)$$

The operators of the displacements of the nonholonomic system of the hoop are

$$Y_0 = \frac{\partial}{\partial t}, \quad Y_1 = \frac{\partial}{\partial \theta} + a \sin\psi \sin\theta \frac{\partial}{\partial \xi} - a \cos\psi \sin\theta \frac{\partial}{\partial \eta} + a \cos\theta \frac{\partial}{\partial \zeta} \tag{3.6}$$
$$Y_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial \psi} - \operatorname{ctg}\theta \frac{\partial}{\partial \varphi}, \qquad Y_3 = \frac{\partial}{\partial \varphi} - a\cos\psi \frac{\partial}{\partial \xi} - a \sin\psi \frac{\partial}{\partial \eta}$$

Here

$$(Y_1, Y_2) = -\operatorname{ctg}\theta\, Y_2 + Y_3, \qquad (Y_2, Y_3) = \frac{a \sin\psi}{\sin\theta} X_4 - \frac{a\cos\psi}{\sin\theta} X_5$$

The kinetic energy T, T° and the force function U are given by [13]

$$T = {}^1/_2\,[(A + a^2)\,\eta_1^2 + A\eta_2^2 + (C + a^2)\,\eta_3^2], \qquad U = -ag\sin\theta$$
$$T^\circ = {}^1/_2\,[(A + a^2\cos^2\theta)\,\eta_1^2 + A\eta_2^2 + C\eta_3^2 + \eta_4^2 + \eta_5^2] \tag{3.7}$$

Equations (1.13) yield

$$(A + a^2)\eta_1' - A\operatorname{ctg}\theta\eta_2^2 + (C + a^2)\eta_2\eta_3 + ag\cos\theta = 0$$
$$A\eta_2' + A\operatorname{ctg}\theta\eta_1\eta_2 - C\eta_1\eta_3 = 0, \qquad (C + a^2)\eta_3' - a^2\eta_1\eta_2 = 0 \tag{3.8}$$

In [13] these equations were derived from the general theorems of dynamics and from Appell's equations.

Substituting (3.5) into the function T° (3.7), we obtain the expression for Θ given in [1],

$$\Theta = {}^1/_2\,[(A + a^2)\eta_1^2 + A\eta_2^2 + (C + a^2)\eta_3^2] \tag{3.9}$$

This expression coincides with the expression for T in (3.7), so that Eqs. (3.14) of [1] also yield (3.8).

The acceleration energy for the hoop is

$$S = {}^1/_2\,[(A + a^2)\eta_1'^2 + A\eta_2'^2 + (C + a^2)\eta_3'^2 + 2\,(A\operatorname{ctg}\theta\eta_2 - C\eta_3)\,(\eta_1\eta_2' - \eta_2\eta_1') -$$
$$- 2a^2\eta_2\,(\eta_1\eta_3' - \eta_3\eta_1')] + \ldots \tag{3.10}$$

Here the dotted lines represent terms not containing η_1', η_2', η_3'.

By virtue of (3.10), Appell's Eqs. (2.3) also yield Eqs. (3.8).

The Cartesian coordinates u_i, v_i, w_i of the ith point of the hoop can be expressed in terms of the chosen variables,

$$u_i = \xi + x_i\,(-\cos\theta\sin\psi\sin\varphi + \cos\psi\cos\varphi) + y_i\,(-\cos\theta\sin\psi\cos\varphi - \cos\theta\sin\varphi) + z_i\sin\theta\sin\psi$$
$$v_i = \eta + x_i\,(\cos\theta\cos\psi\sin\varphi + \sin\psi\cos\varphi) + y_i\,(\cos\theta\cos\psi\cos\varphi - \sin\psi\sin\varphi) - z_i\sin\theta\cos\psi$$
$$w_i = \zeta_i + x_i\sin\theta\sin\varphi + y_i\sin\theta\cos\varphi + z_i\cos\theta \qquad (i = 1, 2, \ldots) \tag{3.11}$$

Here x_i, y_i, z_i are the coordinates of the same ith point in the system whose axes are rigidly attached to the hoop and are its principal axes of inertia.

Commutation relations (2.6) for u_i and v_i, w_i in Hamel's method yield

$$\sum_{s=1}^{5}\left(\frac{d\omega_s}{dt} - \delta\eta_s\right)X_s(u_i) + (\omega_1\eta_2 - \omega_2\eta_1)\,[\operatorname{ctg}\theta\, X_2(u_i) - X_3(u_i)] = 0 \tag{3.12}$$
$$(i = 1, 2, \ldots)$$

By virtue of (3.12), Beltrami Eq. (2.7) for the hoop becomes

$$\sum_{s=1}^{5}\omega_s\left[\frac{d}{dt}\frac{\partial T^\circ}{\partial \eta_s} - X_s(T^\circ + U)\right] - (\omega_1\eta_2 - \omega_2\eta_1)\left(\operatorname{ctg}\theta\frac{\partial T^\circ}{\partial \eta_2} - \frac{\partial T^\circ}{\partial \eta_3}\right) = 0 \tag{3.13}$$

With allowance for the nonholonomic constraints, this equation yields

$$\frac{d}{dt}(A + a^2\cos^2\theta)\,\eta_1 + a\sin\theta\sin\psi\frac{d\eta_4}{dt} - a\sin\theta\cos\psi\frac{d\eta_5}{dt} +$$
$$+ a^2\sin\theta\cos\theta\eta_1^2 - A\operatorname{ctg}\theta\eta_2^2 + C\eta_2\eta_3 + ag\cos\theta = 0 \tag{3.14}$$

$$\frac{d}{dt} A\eta_2 + A \operatorname{ctg} \theta \eta_1 \eta_2 - C\eta_1\eta_2 = 0, \qquad \frac{d}{dt} C\eta_3 - a \cos\psi \frac{d\eta_4}{dt} - a \sin\psi \frac{d\eta_5}{dt} = 0$$

Substituting (3.5) into these expressions, we again have Eqs. (3.8).

Relations (2.12) for u_i (and v_i, w_i) become

$$\sum_{s=1}^{3} \left(\frac{d\omega_s}{dt} - \delta\eta_s\right) Y_s(u_i) + (\omega_1\eta_2 - \omega_2\eta_1)\,[\operatorname{ctg}\theta\, Y_2(u_i) - Y_3(u_i)] +$$

$$+ (\omega_2\eta_3 - \omega_3\eta_2)\left[-\frac{a\sin\psi}{\sin\theta} X_4(u_i) + \frac{a\cos\psi}{\sin\theta} X_5(u_i)\right] = 0 \qquad (i = 1, 2, \ldots) \quad (3.15)$$

Making use of (3.15) without solving them for $d\omega_s / dt - \delta\eta_s$, we can reduce Beltrami Eq. (2.11) to the form

$$\sum_{s=1}^{3} \omega_s \left[\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - Y_s(T + U)\right] - (\omega_1\eta_2 - \omega_2\eta_1)\left(\operatorname{ctg}\theta \frac{\partial T}{\partial \eta_2} - \frac{\partial T}{\partial \eta_3}\right) -$$

$$- (\omega_2\eta_3 - \omega_3\eta_2)\left[-\frac{a\sin\psi}{\sin\theta}\left(\frac{\partial T^\circ}{\partial \eta_4}\right) + \frac{a\cos\psi}{\sin\theta}\left(\frac{\partial T^\circ}{\partial \eta_5}\right)\right] = 0 \qquad (3.16)$$

By (3.7), the latter again gives us Eqs. (3.8).

Wishing to verify conditions (2.16), we obtain

$$Y_1(u_i)\frac{a}{A + a^2} = \frac{\sin\psi}{\sin\theta}, \qquad Y_1(v_i)\frac{a}{A + a_2} = -\frac{\cos\psi}{\sin\theta}, \qquad Y_1(w_i)\frac{a}{A + a^2} = 0$$

$$(i = 1, 2, \ldots)$$

These relations are not fulfilled, since $a \neq 0$.

However, Eqs. (2.14) nevertheless yield the correct equations of motion of the hoop. In fact, solving (3.15) by the Volterra method, we obtain

$$a_{122} = -a_{212} = \operatorname{ctg}\theta, \qquad a_{123} = -a_{213} = -1$$

$$a_{231} = -a_{321} = -\frac{a}{A + a^2}\frac{\sin\psi}{\sin\theta}\sum_{i=1,2\ldots} m_i Y_1(u_i) + \frac{a}{A + a^2}\frac{\cos\psi}{\sin\theta}\sum_{i=1,2\ldots} m_i Y_1(v_i)$$

$$a_{232} = -a_{322} = -\frac{a}{A}\frac{\sin\psi}{\sin\theta}\sum_{i=1,2\ldots} m_i Y_2(u_i) + \frac{a}{A}\frac{\cos\psi}{\sin\theta}\sum_{i=1,2\ldots} m_i Y_2(v_i)$$

$$a_{233} = -a_{323} = -\frac{a}{C + a^2}\frac{\sin\psi}{\sin\theta}\sum_{i=1,2\ldots} m_i Y_3(u_i) + \frac{a}{C + a^2}\frac{\cos\psi}{\sin\theta}\sum_{i=1,2\ldots} m_i Y_3(v_i)$$

Substituting these quantities into (2.14) and recalling that

$$\frac{\partial T^\circ}{\partial \eta_4} = \sum_{i=1,2\ldots} m_i u_i', \qquad \frac{\partial T^\circ}{\partial \eta_5} = \sum_{i=1,2\ldots} m_i v_i'$$

we obtain (3.8), i.e. the equations of motion of the hoop.

Conditions (2.20) and Eqs. (2.21) for the hoop are

$$u_i' = \sum_{s=1}^{3} \eta_s Y_s(u_i), \qquad v_i' = \sum_{s=1}^{3} \eta_s Y_s(v_i), \qquad w_i' = \sum_{s=1}^{3} \eta_s Y_s(w_i) \qquad (i = 1, 2, \ldots)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - \sum_{i=1,2,\ldots} m_i \left[u_i' \frac{dY_s(u_i)}{dt} + v_i' \frac{dY_s(v_i)}{dt} + w_i' \frac{dY_s(w_i)}{dt}\right] = Y_s(U) \qquad (s = 1, 2, 3)$$

By virtue of the relations

$$\frac{dY_s(u_i)}{dt} = \sum_{r=1}^{3} \eta_r Y_r Y_s(u_i), \qquad Y_s(u_i') = \sum_{r=1}^{3} \eta_r Y_s Y_r(u_i) \qquad (s = 1, 2, 3;\ i = 1, 2, \ldots)$$

the above expressions can be rewritten in the form of (2.24),

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_s} - \sum_{r=1}^{3} \eta_r \sum_{i=1,2,\ldots} \left[\frac{\partial T^\circ}{\partial u_i'}(Y_r, Y_s)\, u_i + \frac{\partial T^\circ}{\partial v_i'}(Y_r, Y_s)\, v_i + \right.$$
$$\left. + \frac{\partial T^\circ}{\partial w_i'}(Y_r, Y_s)\, w_i\right] = Y_s(T + U) \qquad (s = 1, 2, 3) \qquad (3.17)$$

Substituting (3.6) and (3.7) into (3.17), we again obtain (3.8). Here

$$T^\circ = \frac{1}{2} \sum_{i=1,2,\ldots} m_i (u_i'^2 + v_i'^2 + w_i'^2)$$

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